

CATEGORICAL HOMOTOPY AND FIBRATIONS

BY

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Let \mathcal{K} be an arbitrary category, and let \mathfrak{M} be any family of its morphisms. It is known [3] that there is a category \mathcal{K}/\mathfrak{M} (the Gabriel-Zisman "category of fractions of \mathcal{K} by \mathfrak{M} ") having the same objects as \mathcal{K} , and a covariant functor $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ which is the identity on objects, such that $\eta(f)$ is invertible in \mathcal{K}/\mathfrak{M} for each $f \in \mathfrak{M}$.

We will use each class \mathfrak{M} to determine a notion of homotopy in \mathcal{K} , by defining two morphisms f, g to be \mathfrak{M} -homotopic if $\eta(f) = \eta(g)$. This notion has the usual properties expected of a homotopy notion; moreover, in the category Top of topological spaces and continuous maps, suitable choices of \mathfrak{M} (for example, \mathfrak{M} = all homotopy equivalences) reveal \mathfrak{M} -homotopy equivalent to the usual notion of homotopy.

Each class \mathfrak{M} determines a notion of fibration in \mathcal{K} . In the category Top , a suitable choice of \mathfrak{M} determines both the usual homotopy and the Hurewicz fibrations; however, different classes \mathfrak{M} may yield the usual notion of homotopy and distinct notions of fibration. More generally, in an arbitrary category \mathcal{K} , it is the given class \mathfrak{M} itself, rather than the homotopy notion induced by \mathfrak{M} , that determines the concept of fibration; from this viewpoint, it turns out, surprisingly, that the notion of a Hurewicz fibration is *not* a homotopy notion. By "reversing arrows," \mathfrak{M} determines also a concept of cofibration; and again there is a splitting: in fact, two classes may determine the same notion of homotopy but distinct notions of cofibration.

In the last section, we introduce the concept of a weak \mathfrak{M} -fibration. This notion does not, in general, possess all the advantages of the previous one; however, it reduces to the previous notion under suitable restrictions on the class \mathfrak{M} . Moreover, there are classes, $\mathfrak{M}, \mathfrak{N}$ in the category Top such that $\{\text{weak } \mathfrak{M}\text{-fibrations}\} = \{\text{Hurewicz fibrations}\}$ and $\{\text{weak } \mathfrak{N}\text{-fibrations}\} = \{\text{Dold fibrations}\}$.

Each covariant functor $\Phi: \mathcal{K} \rightarrow \mathcal{L}$ determines a Φ -homotopy in \mathcal{K} , by choosing $\mathfrak{M} = \{f \mid \Phi(f) \text{ is invertible}\}$. Most of the homotopy notions encountered in various categories turn out to stem from this general construction. For example: if \mathcal{G}^Z is the category of graded groups and degree zero homomorphisms, and if $\pi(X)$ is the total homotopy of a space X , then in the category \mathcal{K} of spaces dominated by CW-complexes the functor $\pi: \mathcal{K} \rightarrow \mathcal{G}^Z$ determines the usual notion of homotopy;

Received by the editors May 9, 1968.

(¹) This research was partially supported by an NSF grant.

if \mathcal{K} is the category of Kan-complexes, then π -homotopy is the same as Kan-homotopy; and if $H: \mathcal{K} \rightarrow \mathcal{G}^Z$ is the total homology functor on the category of chain-complexes, then H -homotopy is the usual chain-homotopy.

In the Appendix, we give a construction of \mathcal{K}/\mathfrak{M} that differs slightly from that of Gabriel-Zisman, and contains slightly more detail; one advantage of this procedure is that some results in [3] are seen to hold without the additional requirement that “ \mathfrak{M} admits a calculus of left fractions”.

1. The quotient category \mathcal{K}/\mathfrak{M} . Let \mathcal{K} be any category, and let \mathfrak{M} be any family of its morphisms. By a quotient category we shall mean a pair $(\mathcal{K}/\mathfrak{M}, \eta)$, where \mathcal{K}/\mathfrak{M} is a category with the same objects as \mathcal{K} and $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ is a covariant functor that preserves objects, having the following two properties:

C1. If $\alpha \in \mathfrak{M}$, then $\eta(\alpha)$ is invertible in \mathcal{K}/\mathfrak{M} .

C2. [Universality]: If $T: \mathcal{K} \rightarrow \mathcal{L}$ is any covariant functor to any category \mathcal{L} such that $T(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$, then there exists a unique covariant functor $\Delta: \mathcal{K}/\mathfrak{M} \rightarrow \mathcal{L}$ such that $T = \Delta \circ \eta$.

1.1. THEOREM. *Let \mathcal{K} be any category, and let \mathfrak{M} be any family of its morphisms. Then a quotient category $(\mathcal{K}/\mathfrak{M}, \eta)$ exists.*

This theorem is mentioned in [3]. However, no detailed construction of $(\mathcal{K}/\mathfrak{M}, \eta)$ having the generality we require appears in the literature, so we will give a proof of 1.1 in the Appendix. It follows directly from our construction, and we will need this in the sequel, that

1.2. PROPOSITION. *Each morphism G in \mathcal{K}/\mathfrak{M} has a factorization $G = \hat{f}_n \circ \hat{\alpha}_n \circ \cdots \circ \hat{f}_1 \circ \hat{\alpha}_1$, where $\hat{f}_i = \eta(f_i)$ for some f_i in \mathcal{K} , and $\hat{\alpha}_i$ is the inverse in \mathcal{K}/\mathfrak{M} for some $\eta(\alpha_i)$, $\alpha_i \in \mathfrak{M}$.*

It is immediate that

1.3. If \mathfrak{M} is the class of all invertible morphisms in \mathcal{K} , or if \mathfrak{M} is the class of all identities in \mathcal{K} , then $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ is an equivalence.

Proof. Since $1: \mathcal{K} \rightarrow \mathcal{K}$ sends each $\alpha \in \mathfrak{M}$ to an invertible morphism, there is, by C2, a factorization $1 = \Delta \circ \eta$ where $\Delta: \mathcal{K}/\mathfrak{M} \rightarrow \mathcal{K}$. This implies $\eta = \eta \circ (\Delta \circ \eta) = (\eta \circ \Delta) \circ \eta$ and since the factorization through η is unique, we must have $\eta \circ \Delta = 1$. Thus $\eta: \mathcal{K} \approx \mathcal{K}/\mathfrak{M}$ is an equivalence⁽²⁾.

Whenever $\mathfrak{M} \subset \mathfrak{M}'$, the canonical projection $\eta': \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}'$ sends each $\alpha \in \mathfrak{M}$ to an invertible morphism, so there is a unique covariant functor $\Delta: \mathcal{K}/\mathfrak{M} \rightarrow \mathcal{K}/\mathfrak{M}'$ such that $\eta' = \Delta \circ \eta$.

⁽²⁾ The unique factorization property characterizes $(\eta, \mathcal{K}/\mathfrak{M})$ up to an equivalence; for the same technique can be used to prove: Let \mathcal{B} be any category, and let $\lambda: \mathcal{K} \rightarrow \mathcal{B}$ be a covariant functor such that $\lambda(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$. Assume that for each category \mathcal{L} , every covariant functor $T: \mathcal{K} \rightarrow \mathcal{L}$ such that $T(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$ factors uniquely through λ . Then there is a unique equivalence $\Delta: \mathcal{K}/\mathfrak{M} \approx \mathcal{B}$ with $\lambda = \Delta \circ \eta$.

Proposition 1.3 is a special case of

1.4. **THEOREM.** *Let $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ be a canonical projection, and let $\overline{\mathfrak{M}} = \{f | \eta(f) \text{ is invertible}\}$. Then there exists a unique equivalence $\Delta: \mathcal{K}/\mathfrak{M} \approx \mathcal{K}/\overline{\mathfrak{M}}$ satisfying $\Delta \circ \eta = \bar{\eta}$. Moreover, $\overline{\mathfrak{M}}$ is the largest class containing \mathfrak{M} for which such an equivalence holds.*

Proof. Since $\mathfrak{M} \subset \overline{\mathfrak{M}}$, we have a unique $\Delta: \mathcal{K}/\mathfrak{M} \rightarrow \mathcal{K}/\overline{\mathfrak{M}}$ with $\bar{\eta} = \Delta \circ \eta$. Since $\eta(f)$ is invertible for each $f \in \overline{\mathfrak{M}}$, there is also a $\bar{\Delta}: \mathcal{K}/\overline{\mathfrak{M}} \rightarrow \mathcal{K}/\mathfrak{M}$ such that $\eta = \bar{\Delta} \circ \bar{\eta}$. Thus, $\eta = \Delta \circ \bar{\Delta} \circ \bar{\eta}$ and $\eta = \bar{\Delta} \circ \Delta \circ \eta$, so by the unique factorization property we find $\Delta \circ \bar{\Delta} = 1$, $\bar{\Delta} \circ \Delta = 1$, therefore Δ is an equivalence. To see that $\overline{\mathfrak{M}}$ is maximal, note that if $\mathfrak{M}_0 \supset \overline{\mathfrak{M}}$, there is only one Δ' such that $\eta_0 = \Delta' \circ \bar{\eta}$, and if $f \in \mathfrak{M}_0 - \overline{\mathfrak{M}}$, then $\eta_0(f)$ is invertible whereas $\bar{\eta}(f)$ is not, so that Δ' cannot be an equivalence.

2. **Homotopy.** Let \mathfrak{M} be an arbitrary class of morphisms in an arbitrary category \mathcal{K} , and let $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ be the canonical projection.

2.1. **DEFINITION.** Two morphisms $f, g: X \rightarrow Y$ in \mathcal{K} are called \mathfrak{M} -homotopic (written $f \cong g \bmod \mathfrak{M}$) if $\eta(f) = \eta(g)$.

This has the standard properties required of a homotopy notion:

2.2. **THEOREM.** (a) \mathfrak{M} -homotopy is an equivalence relation in each $\mathcal{K}(X, Y)$.

(b) Let $f_0, f_1: X \rightarrow Y$ be \mathfrak{M} -homotopic. If $g: W \rightarrow X$ and $h: Y \rightarrow Z$ are any two morphisms, then $f_0 \circ g \cong f_1 \circ g \bmod \mathfrak{M}$ and $h \circ f_0 \cong h \circ f_1 \bmod \mathfrak{M}$.

(c) Each invertible morphism in \mathcal{K} is an \mathfrak{M} -homotopy equivalence⁽³⁾.

(d) If $f, g: X \rightarrow Y$ have an equalizer (or a coequalizer) $c \in \mathfrak{M}$, then $f \cong g \bmod \mathfrak{M}$.

Proof. (a)–(c) are trivial, since η is a covariant functor. For (d): Assume, say, $c \circ f = c \circ g$; then $\eta(c)\eta(f) = \eta(c)\eta(g)$ and, since $c \in \mathfrak{M}$ so that $\eta(c)$ is invertible, we conclude that $\eta(f) = \eta(g)$.

Denote the \mathfrak{M} -homotopy class of $f \in \mathcal{K}(X, Y)$ by $[f]_{\mathfrak{M}}$ and the set of all \mathfrak{M} -homotopy classes in $\mathcal{K}(X, Y)$ by $[X, Y]_{\mathfrak{M}}$. From 2.2 it follows in the usual way that (1) there is a category $\mathcal{K}(\mathfrak{M})$ whose objects are those of \mathcal{K} and whose morphisms are the \mathfrak{M} -homotopy classes, and (2) the function $p: \mathcal{K} \rightarrow \mathcal{K}(\mathfrak{M})$ which is the identity on objects and sends each f to $[f]_{\mathfrak{M}}$, is a covariant functor⁽⁴⁾.

This method of defining a homotopy notion in a category is reasonable, in that by a suitable choice (in fact, many choices) of \mathfrak{M} , one gets the usual notion of homotopy in the category **Top** of all topological spaces and continuous maps. This will follow from the general

⁽³⁾ Precisely, $f: X \rightarrow Y$ is an \mathfrak{M} -homotopy equivalence if there exists a $g: Y \rightarrow X$ such that $f \circ g \cong 1 \bmod \mathfrak{M}$ and $g \circ f \cong 1 \bmod \mathfrak{M}$.

⁽⁴⁾ In particular, if $c: A \rightarrow A$ is such that $c \circ f = c \circ g$ for every $f, g: Z \rightarrow A$, then whenever $c \in \mathfrak{M}$, the set $[Z, A]_{\mathfrak{M}}$ consists of a single element.

2.3. THEOREM. Let \mathcal{K} be a category and \sim an equivalence relation in each $\mathcal{K}(A, B)$ such that the transformation $h: \mathcal{K} \rightarrow \mathcal{K}_H$ taking each object A to itself and each f to its \sim -class is a covariant functor.

Assume that, if $f_0 \sim f_1: X \rightarrow Y$ then there exists an object $I_X \in \mathcal{K}$, and morphisms, $r: I_X \rightarrow X$, $i_0, i_1: X \rightarrow I_X$, $F: I_X \rightarrow Y$, such that $r \circ i_j = 1$ and $f_j = F \circ i_j$ ($j=0, 1$).

Finally, let \mathfrak{M} be a class of morphisms in \mathcal{K} and $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ the projection. Then

- (a) If $\eta(r)$ is invertible in \mathcal{K}/\mathfrak{M} , then $f_0 \sim f_1$ implies $f_0 \cong f_1 \bmod \mathfrak{M}$.
- (b) If $h(\alpha)$ is invertible in \mathcal{K}_H for each $\alpha \in \mathfrak{M}$, then $f_0 \cong f_1 \bmod \mathfrak{M}$ implies $f_0 \sim f_1$.

Proof. (a) Assume $f_0 \sim f_1$. Since $\eta(r) \circ \eta(i_j) = 1$ ($j=0, 1$) and $\eta(r)$ is invertible, we find that each $\eta(i_j)$ is invertible, and that $\eta(i_0) = \eta(r)^{-1} = \eta(i_1)$. Thus $\eta(f_0) = \eta(F)\eta(i_0) = \eta(F)\eta(i_1) = \eta(f_1)$ so $f_0 \cong f_1 \bmod \mathfrak{M}$.

(b) Because of C2, the functor h has a factorization $h = \Delta \circ \eta$; therefore $\eta(f_0) = \eta(f_1)$ implies $h(f_0) = h(f_1)$, i.e., that $f_0 \sim f_1$. This completes the proof.

Let $\mathcal{K} = \text{Top}$ and \sim the usual homotopy in Top ; then by taking \mathfrak{M} to be any one of the four classes:

- \mathfrak{U} = all homotopy equivalences,
- \mathfrak{B} = all maps $r: X \times I \rightarrow X$, where $r(x, t) = x$,
- \mathfrak{S} = all maps $i: X \rightarrow X \times I$, where $i(x) = (x, 0)$,
- \mathfrak{X} = all inclusion maps $j: W \rightarrow X$, where $j(W)$ is a zero-set⁽⁵⁾ and a strong deformation retract of X , it follows immediately from 2.3 that \mathfrak{M} -homotopy is exactly the same as the usual homotopy.

We shall say that two classes $\mathfrak{M}, \mathfrak{N}$ in a category \mathcal{K} determine the same notion of homotopy, and write $[\mathfrak{M}] = [\mathfrak{N}]$, whenever $f \cong g \bmod \mathfrak{M}$ if and only if $f \cong g \bmod \mathfrak{N}$. As the examples in Top show, distinct classes may yield the same homotopy notion. We now examine briefly some questions that arise, such as: to find conditions assuring that the \mathfrak{M} -homotopy equivalences determine the same homotopy notion as \mathfrak{M} itself.

Let \mathfrak{M} be fixed, let $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ be the canonical map, and let

$$\begin{aligned}\bar{\mathfrak{M}} &= \{f | \eta(f) \text{ is invertible}\}, \\ E(\mathfrak{M}) &= \{f | \eta(f) \text{ is invertible and has some } \eta(g) \text{ as inverse}\} \\ &= \{f | f \text{ is an } \mathfrak{M}\text{-homotopy equivalence}\}.\end{aligned}$$

It is clear that, if $\mathfrak{M} \subset \mathfrak{N}$, then $\bar{\mathfrak{M}} \subset \bar{\mathfrak{N}}$ and $E(\mathfrak{M}) \subset E(\mathfrak{N})$. The general relation of these concepts is

2.4. (a) $[\mathfrak{M}] = [\bar{\mathfrak{M}}]$ and $E(\mathfrak{M}) = E(\bar{\mathfrak{M}}) \subset \bar{\mathfrak{M}}$; (b) $\bar{\mathfrak{M}} = \bar{\mathfrak{N}} \Rightarrow [\mathfrak{M}] = [\mathfrak{N}] \Rightarrow E(\mathfrak{M}) = E(\mathfrak{N})$.

Proof. (a) According to 1.4, there are equivalences $\Delta: \mathcal{K}/\mathfrak{M} \approx \mathcal{K}/\bar{\mathfrak{M}}$, $\bar{\Delta}: \mathcal{K}/\bar{\mathfrak{M}} \approx \mathcal{K}/\mathfrak{M}$ such that $\bar{\eta} = \Delta \circ \eta$, $\eta = \bar{\Delta} \circ \bar{\eta}$. This implies that $\eta(f) = \eta(g)$ if and only if

⁽⁵⁾ I.e., there exists a continuous $\phi: X \rightarrow I$ with $\phi^{-1}(0) = j(W)$.

$\bar{\eta}(f) = \bar{\eta}(g)$, so that $[\mathfrak{M}] = [\bar{\mathfrak{M}}]$. It also implies that $E(\mathfrak{M}) = E(\bar{\mathfrak{M}})$; and the definition shows $E(\mathfrak{M}) \subset \bar{\mathfrak{M}}$.

(b) If $\bar{\mathfrak{M}} = \bar{\mathfrak{N}}$ then obviously $[\bar{\mathfrak{M}}] = [\bar{\mathfrak{N}}]$ and, from (a), also $[\mathfrak{M}] = [\mathfrak{N}]$. Now let $f \in E(\mathfrak{M})$; then there is a g with $f \circ g \cong 1 \bmod \mathfrak{M}$, $g \circ f \cong 1 \bmod \mathfrak{M}$; since $[\mathfrak{M}] = [\mathfrak{N}]$, we find $f \circ g \cong 1 \bmod \mathfrak{N}$, $g \circ f \cong 1 \bmod \mathfrak{N}$ and so $f \in E(\mathfrak{N})$. Similarly, $f \in E(\mathfrak{N})$ implies $f \in E(\mathfrak{M})$ and the proof is complete.

In general, none of the implications in 2.4(b) are reversible.

(A) Let \mathcal{K} consist of two objects A, B together with the identity morphisms and a single morphism $f: A \rightarrow B$. Let $\mathfrak{M} = \{f, 1_A, 1_B\}$ and $\mathfrak{N} = \{1_A, 1_B\}$; then $[\mathfrak{M}] = [\mathfrak{N}]$ yet $\bar{\mathfrak{M}} \neq \bar{\mathfrak{N}}$ (see 1.3).

(B) Let \mathcal{K} consist of three intervals $A_i = [0, r]$, $r = 1, 2, 3$, with the identity maps, the inclusion maps $j_{rs}: A_r \rightarrow A_s$, $r < s$, and the map $h: A_2 \rightarrow A_3$ given by $h(x) = x$, $0 \leq x \leq 1$, $h(x) = 1$, $x > 1$, so that $j_{13} = j_{23} \circ j_{12} = h \circ j_{12}$. Let $\mathfrak{M} = \{1_1, 1_2, 1_3, j_{12}\}$ and $\mathfrak{N} = \{1_1, 1_2, 1_3\}$; then $[\mathfrak{M}] \neq [\mathfrak{N}]$ since h is \mathfrak{M} -homotopic to j_{23} ; however $E(\mathfrak{M}) = E(\mathfrak{N}) = \mathfrak{N}$ since there is no map in \mathcal{K} going backwards. Note that this shows also that $[\mathfrak{M}] \neq [\mathfrak{N}] = [E(\mathfrak{M})]$.

2.5. DEFINITION. The class \mathfrak{M} is called stable if $E(\mathfrak{M}) = \bar{\mathfrak{M}}$; i.e., if the class of $\bar{\mathfrak{M}}$ -homotopy equivalences is exactly $\bar{\mathfrak{M}}$.

If \mathfrak{M} is stable, then $E(\mathfrak{M})$ and \mathfrak{M} determine the same notion of homotopy, since $[E(\mathfrak{M})] = [\bar{\mathfrak{M}}] = [\mathfrak{M}]$. Furthermore, distinct stable classes determine distinct notions of homotopy, because, from 2.4(b) and stability, we have

2.6. If $\mathfrak{M}, \mathfrak{N}$ are stable classes, then $\bar{\mathfrak{M}} = \bar{\mathfrak{N}} \Leftrightarrow [\mathfrak{M}] = [\mathfrak{N}] \Leftrightarrow E(\mathfrak{M}) = E(\mathfrak{N})$.

The stable classes are characterized by

2.7. THEOREM. A class \mathfrak{M} in \mathcal{K} is stable if and only if $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ is epic⁽⁶⁾.

Proof. "If": We need show only that $\bar{\mathfrak{M}} \subset E(\mathfrak{M})$. Let $f \in \bar{\mathfrak{M}}$; then $\eta(f)$ has an inverse G in \mathcal{K}/\mathfrak{M} ; since η is epic, $G = \eta(g)$ for some g in \mathcal{K} , so $f \in E(\mathfrak{M})$. "Only if": We have seen in 1.2 that each given G in \mathcal{K}/\mathfrak{M} has a factorization as $G = f_n \circ \hat{\alpha}_n \circ \cdots \circ f_1 \circ \hat{\alpha}_1$, where each $f_i = \eta(f_i)$ for some f_i in \mathcal{K} , and each $\hat{\alpha}_i$ is the inverse of some $\eta(\alpha_i)$. In particular, each $\alpha_i \in \bar{\mathfrak{M}} = E(\mathfrak{M})$, so $\eta(\alpha_i)$ has some $\eta(\beta_i)$ for inverse and, since inverses are unique, $\hat{\alpha}_i = \eta(\beta_i)$. Thus, $G = \eta[f_n \circ \beta_n \circ \cdots \circ f_1 \circ \beta_1]$ and η is epic.

2.8. COROLLARY. Let \mathfrak{M} be any class. Then there exists a unique maximal stable class $\hat{\mathfrak{M}} \subset \mathfrak{M}$ with the following property: if $\mathfrak{N} \subset \mathfrak{M}$ is stable, then $\mathfrak{N} \subset \hat{\mathfrak{M}}$.

Proof. In view of our constructions, the empty class $\emptyset \subset \mathfrak{M}$ is (1.3) clearly stable. It is therefore enough to show that the union of any family of stable classes is stable. Let $\{\mathfrak{M}_\beta \mid \beta \in B\}$ be the family of all stable classes in \mathfrak{M} , so that each $\eta_\beta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}_\beta$ is epic; letting $\hat{\mathfrak{M}} = \bigcup \mathfrak{M}_\beta$, we will prove that $\eta: \mathcal{K} \rightarrow \mathcal{K}/\hat{\mathfrak{M}}$ is also epic. Each morphism G in $\mathcal{K}/\hat{\mathfrak{M}}$ has a factorization $G = f_n \circ \hat{\alpha}_n \circ \cdots \circ f_1 \circ \hat{\alpha}_1$,

⁽⁶⁾ I.e., for each A, B and $f \in \text{hom}(A, B)$, there exists an $g: A \rightarrow B$ with $\eta(f) = \eta(g)$. Note that the stability of \mathfrak{M} implies that each $\text{hom}(A, B)$ is in fact a set, cf. footnote 14, 15.

where each $f_i = \eta(f_i)$ for some f_i in \mathcal{K} . Each α_i lies in some \mathfrak{M}_i so there is a β_i in \mathcal{K} with $\eta(\beta_i) = \alpha_i$ in $\mathcal{K}/\mathfrak{M}_i$. Since (1.4) there is an $\hat{\eta}_i: \mathcal{K}/\mathfrak{M}_i \rightarrow \mathcal{K}/\hat{\mathfrak{M}}$ with $\eta = \hat{\eta}_i \circ \eta_i$, it follows that $\eta(\beta_i) = \hat{\alpha}_i$ in $\mathcal{K}/\hat{\mathfrak{M}}$ and therefore that $\eta(f_n \circ \beta_n \circ \dots \circ f_1 \circ \beta_1) = G$. This completes the proof.

The maximal stable class $\hat{\mathfrak{M}}$ may not give the same notion of homotopy as \mathfrak{M} . In fact, there are notions of \mathfrak{M} -homotopy that cannot be obtained from any stable class: in example (B) above, $\hat{\mathfrak{M}} = \mathfrak{M}$ and, indeed, \mathfrak{M} -homotopy cannot be obtained from a stable class. Moreover, $E(\mathfrak{M}) \neq E(\hat{\mathfrak{M}})$ in general (cf. Appendix A5).

The machinery developed so far can be applied to any covariant functor $\Phi: \mathcal{K} \rightarrow \mathcal{L}$ to introduce into \mathcal{K} a notion of homotopy induced (or adapted to) the functor Φ .

2.9. DEFINITION. Let $\Phi: \mathcal{K} \rightarrow \mathcal{L}$ be any covariant functor. Let

$$\mathfrak{M}(\Phi) = \{f \text{ in } \mathcal{K} \mid \Phi(f) \text{ is invertible}\}.$$

The homotopy in \mathcal{K} determined by the class $\mathfrak{M}(\Phi)$ is called Φ -homotopy.

The category $\mathcal{K}/\mathfrak{M}(\Phi)$ is written \mathcal{K}^Φ and is essentially the category first studied by Bauer in [1]. Observe that, because of C2, the functor Φ has a unique factorization $\Phi = \Delta \circ \eta$ through \mathcal{K}^Φ . It follows from this that $\mathfrak{M}(\Phi) = [\mathfrak{M}(\Phi)]^-$ always: for, if $f \in [\mathfrak{M}(\Phi)]^-$, then $\eta(f)$ is invertible, therefore so also is $\Delta(\eta(f)) = \Phi(f)$, consequently $f \in \mathfrak{M}(\Phi)$; thus, $[\mathfrak{M}(\Phi)]^- \subset \mathfrak{M}(\Phi)$ and the opposite inclusion is trivial.

We give some applications of Φ -homotopy.

(1) Let $\Phi: \text{Top} \rightarrow \text{Ens}$ be the forgetful functor. Then f is Φ -homotopic to g if and only if $f = g$. For, consider the factorization $\Phi = \Delta \circ \eta$ through \mathcal{K}^Φ . If $\eta(f) = \eta(g)$, then $\Phi(f) = \Phi(g)$ so that $f = g$ as maps of sets; and the converse is trivially true. The class $\mathfrak{M} = \mathfrak{M}(\Phi)$ consists of all bijective continuous maps, whereas $E(\mathfrak{M})$ is the class of all bijective bicontinuous maps (i.e., homeomorphisms).

(2) Let \mathcal{K} be the category of spaces dominated by CW-complexes, let \mathcal{G}^Z be the category of graded groups and degree zero homomorphisms, and let $\pi: \mathcal{K} \rightarrow \mathcal{G}^Z$ be the total homotopy functor. Then two maps are π -homotopic if and only if they are homotopic. For, we apply 2.3 because (a) each projection $r: X \times I \rightarrow X$ is such that $\pi(r)$ is invertible, therefore $r \in \mathfrak{M}(\pi)$, consequently $\eta(r)$ is invertible, and (b) if $f \in \mathfrak{M}(\pi)$, i.e., if $\pi(f)$ is invertible, then, by Whitehead's theorem [6] f is a homotopy equivalence and therefore $h(f)$ is invertible. Note that, again using Whitehead's theorem, the class $\mathfrak{M}(\pi)$ is stable, so that $\eta: \mathcal{K} \rightarrow \mathcal{K}^\pi$ is epic. Similarly, if \mathcal{K} is the category of Kan-complexes, then π -homotopy in \mathcal{K} is exactly Kan-homotopy.

(3) Let \mathcal{K} be the category of all simply connected CW-complexes, and $H: \mathcal{K} \rightarrow \mathcal{G}^Z$ the total homology functor. Exactly the same considerations as in (2) reveal H -homotopy to be the usual homotopy. Using $H: \mathcal{K} \rightarrow \mathcal{G}^Z$ on the category of css complexes yields a homotopy notion extending the usual one for Kan-complexes. And on the category \mathcal{K} of chain-complexes, H -homotopy yields the usual notion of chain-homotopy.

Thus, the various notions of homotopy encountered, and usually defined

differently, in various categories turn out to be essentially the homotopy relation determined in the above manner by a suitable covariant functor. For arbitrary contravariant functors, one turns to the dual category.

3. Fibrations. Each class \mathfrak{M} of morphisms in a category \mathcal{K} determines a concept of fibration in \mathcal{K} :

3.1. DEFINITION. A morphism $p: E \rightarrow B$ in \mathcal{K} is called an \mathfrak{M} -fibration if for each diagram

$$\begin{array}{ccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E \\ & & \searrow f & & \downarrow p \\ & & & & B \end{array}$$

in which $\mu \in \mathfrak{M}$ and $p \circ g \circ \mu = f \circ \mu$, there exists a $g': X \rightarrow E$ in \mathcal{K} with $g \circ \mu = g' \circ \mu$ and $p \circ g' = f$.

We are requiring simply that every triangle involving p which can be equalized by a morphism belonging to \mathfrak{M} , can be made itself commutative. Note that if \mathfrak{M} -homotopy is used in \mathcal{K} , the conditions $p \circ g \circ \mu = f \circ \mu$ and $g \circ \mu = g' \circ \mu$ imply, by 2.2(d), that $p \circ g \cong f \text{ mod } \mathfrak{M}$ and $g \cong g' \text{ mod } \mathfrak{M}$. The relations between \mathfrak{M} -homotopy and \mathfrak{M} -fibration will be considered after we verify that the class of \mathfrak{M} -fibrations has the properties usually required of fibrations.

3.2. THEOREM. Let \mathfrak{M} be a fixed class of morphisms in \mathcal{K} . Then:

- (a) If $p: E \rightarrow B$ is invertible, then p is an \mathfrak{M} -fibration.
- (b) If $p_1: E_1 \rightarrow E_0$ and $p_0: E_0 \rightarrow B$ are \mathfrak{M} -fibrations, so also is $p_0 \circ p_1: E_1 \rightarrow B$.
- (c) If $p: E \rightarrow B$ is an \mathfrak{M} -fibration, and if

$$\begin{array}{ccc} E' & \xrightarrow{\tilde{h}} & E \\ p' \downarrow & & \downarrow p \\ B' & \xrightarrow{h} & B \end{array}$$

is a cartesian diagram⁽⁷⁾, then p' is also an \mathfrak{M} -fibration; and h can be lifted into E if and only if p' has a section⁽⁸⁾.

Proof. (a) is trivial.

⁽⁷⁾ In the terminology of Mitchell [4] the indicated diagram would be called a pullback of p, h .

⁽⁸⁾ I.e., an $s: B' \rightarrow E'$ such that $p' \circ s = 1$.

(b) Given

$$\begin{array}{ccccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E_1 & \xrightarrow{p_1} & E_0 \\ & & & \searrow f & & & \downarrow p_0 \\ & & & & & & B \end{array}$$

with $p_0 \circ p_1 \circ g \circ \mu = f \circ \mu$ and $\mu \in \mathfrak{M}$, there exists first a $G: X \rightarrow E_0$ with $p_0 \circ G = f$, $G \circ \mu = p_1 \circ g \circ \mu$; and then a $g': X \rightarrow E_1$ with $p_1 \circ g' = G$, $g' \circ \mu = g \circ \mu$; since $p_0 \circ p_1 \circ g \circ \mu = p_0 \circ G = f$, the required morphism is g' .

(c) We are given

$$\begin{array}{ccccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E' & \xrightarrow{\tilde{h}} & E \\ & & \searrow f & & \downarrow p' & & \downarrow p \\ & & & & B' & \xrightarrow{h} & B \end{array}$$

where the square is cartesian and $f \circ \mu = p' \circ g \circ \mu$. Thus, $h \circ f \circ \mu = p \circ \tilde{h} \circ g \circ \mu$ so, because p is an \mathfrak{M} -fibration, there is a $G: X \rightarrow E$ such that $G \circ \mu = \tilde{h} \circ g \circ \mu$ and $p \circ G = h \circ f$. Because the square is cartesian and $p \circ G = h \circ f$, there exists a unique $g': X \rightarrow E'$ such that $\tilde{h} \circ g' = G$ and $p' \circ g' = f$. Since also $\tilde{h} \circ (g' \circ \mu) = G \circ \mu = \tilde{h} \circ (g \circ \mu)$ and $p' \circ (g \circ \mu) = f \circ \mu = p' \circ (g' \circ \mu)$, the uniqueness of a morphism $W \rightarrow E'$ satisfying these two conditions in a cartesian diagram shows $g' \circ \mu = g \circ \mu$ and therefore we find that p' is an \mathfrak{M} -fibration. The second part is proved in a similar manner.

We now examine the notion of \mathfrak{M} -fibration in Top , using the classes, and the symbols for those classes, that are listed following Theorem 2.3. We have

3.3 THEOREM. *Let $\mathfrak{M} = \mathfrak{F}$ in Top . Then \mathfrak{F} -homotopy is the classical notion, and $p: E \rightarrow B$ is a \mathfrak{F} -fibration if and only if it is a Hurewicz fibration.*

Proof. The first part was established in 2.3. Let now $p: E \rightarrow B$ be a \mathfrak{F} -fibration; we show it has the covering homotopy property, i.e., that it is a Hurewicz fibration. We start with the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ i \downarrow & & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

where $i(x) = (x, 0)$. Let $r: X \times I \rightarrow X$ be the map $r(x, t) = x$; we then have

$$\begin{array}{ccccc} X & \xrightarrow{i} & X \times I & \xrightarrow{f \circ r} & E \\ & & \searrow F & & \downarrow p \\ & & & & B \end{array}$$

where $F \circ i = p \circ f = p \circ f \circ r \circ i$. Since p is a \mathfrak{F} -fibration, there is an $\bar{F}: X \times I \rightarrow E$ with $p \circ \bar{F} = F$ and $\bar{F} \circ i = f \circ r \circ i = f$, so that \bar{F} is a homotopy of f covering F , and therefore p is a Hurewicz fibration.

Conversely, let $p: E \rightarrow B$ be a Hurewicz fibration. The morphisms in \mathfrak{F} being simply the maps $i: X \rightarrow X \times I$, we consider any diagram

$$\begin{array}{ccccc} X & \xrightarrow{i} & X \times I & \xrightarrow{f} & E \\ & & & \searrow F & \downarrow p \\ & & & & B \end{array}$$

in which $p \circ f \circ i = F \circ i$; since p is a Hurewicz fibration, there is a homotopy \bar{F} of $f \circ i$ covering F , so that $p \circ \bar{F} = F$ and $f \circ i = \bar{F} \circ i$. Thus, p is a \mathfrak{F} -fibration, and the proof is complete.

If $\mathfrak{M} \subset \mathfrak{N}$, it is clear⁽⁹⁾ that $\{\mathfrak{N}\text{-fibrations}\} \subset \{\mathfrak{M}\text{-fibrations}\}$. Moreover,

(a) Two classes may determine the same notion of homotopy, but distinct notions of fibration.

Example in Top: We have seen that \mathfrak{F} determines the same homotopy notion as \mathfrak{H} , that is, the usual notion of homotopy. However, every continuous map is a \mathfrak{F} -fibration: for, clearly, $\mathfrak{F} \subset$ (all continuous surjections), and, if μ is surjective, the condition $p \circ g \circ \mu = f \circ \mu$ of Definition 3.1 implies that $p \circ g = f$; thus, every continuous $p: E \rightarrow B$ belongs to $\{\text{surjective map fibrations}\} \subset \{\mathfrak{F}\text{-fibrations}\}$.

(b) Two classes may determine the same notion of fibration but distinct notions of homotopy.

Example in Top: Let \mathfrak{M} = class of all identity maps $1_A: A \rightarrow A$. Then every continuous map is an \mathfrak{M} - and also a \mathfrak{F} -fibration; yet \mathfrak{F} -homotopy is the usual notion of homotopy, whereas we have seen that \mathfrak{M} -homotopy is simply equality.

From this viewpoint, the notions of homotopy and of fibration are independent. In the category Top, the Hurewicz fibrations appear as a concept dependent on a particular class of morphisms that happens to yield the usual homotopy notion, rather than as a concept dependent on the homotopy notion itself⁽¹⁰⁾.

Returning to 3.1, we shall establish a simple criterion for two classes to determine the same notion of fibration. This is based on

3.4. THEOREM. *Let $\mathfrak{M}, \mathfrak{Q}$, be two classes of morphisms in \mathcal{K} . Assume that for each $\mu: W \rightarrow X$, $\mu \in \mathfrak{M}$, there exist a $\lambda: Y \rightarrow Z$, $\lambda \in \mathfrak{Q}$ and morphisms in \mathcal{K} such that*

⁽⁹⁾ We denote the class of \mathfrak{M} -fibrations by $\{\mathfrak{M}\text{-fibrations}\}$.

⁽¹⁰⁾ The class \mathcal{H} of Hurewicz fibrations does not determine the usual homotopy: in fact, since every constant map $f: E \rightarrow e$ is a Hurewicz fibration, it follows (cf. footnote 4) that for each X, Y , all $f, g: X \rightarrow Y$ are \mathcal{H} -homotopic.

$$\begin{array}{ccccc}
 W & \xrightarrow{i} & Y & \xrightarrow{s} & W \\
 \mu \downarrow & & \lambda \downarrow & & \downarrow \mu \\
 X & \xrightarrow{j} & Z & \xrightarrow{k} & X
 \end{array}$$

is commutative and $k \circ j = 1$. Then $\{\mathfrak{L}\text{-fibrations}\} \subset \{\mathfrak{M}\text{-fibrations}\}$.

Proof. Let $p: E \rightarrow B$ be an \mathfrak{L} -fibration. Given a diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{\mu} & X & \xrightarrow{g} & E \\
 & & \searrow f & & \downarrow p \\
 & & & & B
 \end{array}$$

with $p \circ g \circ \mu = f \circ \mu$ and $\mu \in \mathfrak{M}$, we find that $p \circ g \circ k \circ \lambda = p \circ g \circ \mu \circ s = f \circ \mu \circ s = f \circ k \circ \lambda$ so, since p is an \mathfrak{L} -fibration, there is a $\Gamma: Z \rightarrow E$ with $p \circ \Gamma = f \circ k$ and $\Gamma \circ \lambda = g \circ k \circ \lambda$. Letting $G = \Gamma \circ j$, we find $p \circ G = p \circ \Gamma \circ j = f \circ k \circ j = f$, and $G \circ \mu = \Gamma \circ j \circ \mu = g \circ k \circ \lambda \circ i = g \circ k \circ j \circ \mu = g \circ \mu$, so that p is an \mathfrak{M} -fibration.

As an application,

3.5. In Top, $\{\mathfrak{X}\text{-fibrations}\} = \{\text{Hurewicz fibrations}\}$.

Proof. Clearly, $\mathfrak{X} \subset \mathfrak{X}$, so $\{\mathfrak{X}\text{-fibrations}\} \subset \{\mathfrak{X}\text{-fibrations}\}$. For the converse, let $\mu: W \rightarrow X$, $\mu \in \mathfrak{X}$, be given; we identify W with a subset of X in order to cut down excessive notation, and let $\Phi: r \cong 1$ be a strong deformation retraction. Choose $\phi: X \rightarrow I$ vanishing exactly on W , and define $\bar{\Phi}: X \times I \rightarrow X$ by

$$\begin{aligned}
 \bar{\Phi}(x, t) &= \Phi(x, t/\phi(x)), & x \notin W, \\
 &= \Phi(x, 1), & x \in W,
 \end{aligned}$$

which is easily verified to be continuous. Then the diagram

$$\begin{array}{ccccc}
 W & \xrightarrow{\mu} & X & \xrightarrow{r} & W \\
 \mu \downarrow & & \downarrow \lambda & & \downarrow \mu \\
 X & \xrightarrow{j} & X \times I & \xrightarrow{\bar{\Phi}} & X
 \end{array}$$

where r =retraction, μ =inclusion, $\lambda(x)=(x, 0)$ and $j(x)=(x, \phi(x))$ is easily seen to be commutative. Thus, $\{\mathfrak{X}\text{-fibrations}\} \subset \{\mathfrak{X}\text{-fibrations}\}$ and, with 3.3, the theorem is proved.

We recall that, in Top, a fibration is called regular whenever a covering homotopy can always be chosen stationary at all the points where the given homotopy is stationary⁽¹¹⁾. Let $\tilde{\mathfrak{X}}$ be the family of all inclusions $\mu: A \rightarrow X$, where A is a

⁽¹¹⁾ An example of a nonregular Hurewicz fibration, as well as a general condition on a topological space B that assures the regularity of every Hurewicz fibration over B , is given in [5].

strong deformation retract of X , and let $\tilde{\mathfrak{B}}$ be the class of all maps $\lambda_B: X \rightarrow X_B = (X \times I) \cup_{r|B \times I} B$, $r \in \mathfrak{B}$, where $B \subset X$ is arbitrary, and $\lambda_B = \rho_B \circ i$ with $i \in \mathfrak{I}$ and $\rho_B: X \times I \rightarrow X_B$ the identification. Clearly, $\mathfrak{X} \subset \tilde{\mathfrak{X}}$ and $\mathfrak{I} \subset \mathfrak{B} \subset \tilde{\mathfrak{X}}$, so that every $\tilde{\mathfrak{X}}$ - and every $\tilde{\mathfrak{B}}$ -fibration is a Hurewicz fibration.

3.7. THEOREM⁽¹²⁾. In Top $\{\tilde{\mathfrak{X}}\text{-fibrations}\} = \{\tilde{\mathfrak{B}}\text{-fibrations}\} = \{\text{regular Hurewicz fibrations}\}$.

Proof. The proof that $\{\tilde{\mathfrak{B}}\text{-fibrations}\} = \{\text{regular Hurewicz fibrations}\}$ is a repetition of that for 3.3, with $X \times I$ replaced by suitable X_B and \mathfrak{I} replaced by $\tilde{\mathfrak{B}}$. Because $\tilde{\mathfrak{B}} \subset \tilde{\mathfrak{X}}$, we have $\{\tilde{\mathfrak{X}}\text{-fibrations}\} \subset \{\tilde{\mathfrak{B}}\text{-fibrations}\}$. To prove the converse inclusion, let $\mu: A \rightarrow X$, $\mu \in \tilde{\mathfrak{X}}$, be given, let $\Phi: r \simeq 1$, be a strong deformation retraction of X onto A ; then a continuous $\bar{\Phi}: X_A \rightarrow X$ such that $\bar{\Phi} \circ \rho_A = \Phi$ exists. Let $i_1: X \rightarrow X \times I$ be the map $x \rightarrow (x, 1)$; then the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\mu} & X & \xrightarrow{r} & A \\ \mu \downarrow & & \downarrow \lambda_A & & \downarrow \mu \\ X & \xrightarrow{j} & X_A & \xrightarrow{\bar{\Phi}} & X \end{array}$$

with $j = \rho_A \circ i_1$, is commutative and $\bar{\Phi} \circ j = 1$. Thus, by 3.5, the proof is complete.

As in the previous section, a covariant functor $\Phi: \mathcal{K} \rightarrow \mathcal{L}$ gives rise to a notion of Φ -fibration, i.e., $\mathfrak{M}(\Phi)$ -fibration. Observe that, if $\pi: \mathcal{K} \rightarrow \mathcal{G}^Z$ is the total homotopy functor on the category of spaces dominated by CW-complexes, then the corresponding notion of Φ -fibration is not that of Hurewicz fibration, but of a certain subset: for, $\mathfrak{M}(\Phi) = \text{set of all homotopy equivalences}$ and $\mathfrak{I} \subset \mathfrak{M}(\Phi)$ so that $\{\mathfrak{M}(\Phi)\text{-fibrations}\} \subset \{\mathfrak{I}\text{-fibrations}\} = \{\text{Hurewicz fibrations}\}$.

4. Cofibrations.

4.1. DEFINITION. Let \mathfrak{M} be a class of morphisms in a category \mathcal{K} . A morphism $j: B \rightarrow E$ in \mathcal{K} is called an \mathfrak{M} -cofibration if for each diagram

$$\begin{array}{ccccc} E & \xrightarrow{g} & X & \xrightarrow{\mu} & W \\ j \uparrow & \nearrow f & & & \\ B & & & & \end{array}$$

in which $\mu \in \mathfrak{M}$ and $\mu \circ g \circ j = \mu \circ f$, there exists a $g': E \rightarrow X$ with $g' \circ j = f$ and $\mu \circ g' = \mu \circ g$.

This concept depends on the class \mathfrak{M} itself rather than on the homotopy notion that \mathfrak{M} determines. In fact, the corresponding homotopy notions may be the same, and the associated cofibrations different, since

4.2. THEOREM. In the category Top ,

⁽¹²⁾ For $\tilde{\mathfrak{X}}$ -fibrations, this theorem was suggested by the referee.

- (a) $\{\mathfrak{F}\text{-cofibrations}\} = \{\mathfrak{X}\text{-cofibrations}\} = \text{all morphisms in Top};$
 (b) $\{\mathfrak{B}\text{-cofibrations}\} = \text{class of all } j: B \rightarrow E \text{ such that every continuous } \phi: B \rightarrow I \text{ has a factorization } \phi = \psi \circ j \text{ where } \psi: E \rightarrow I.$

Proof. (a) is trivial. (b) Assume that $j: B \rightarrow E$ is a \mathfrak{B} -cofibration. Let $\phi: B \rightarrow I$ be given and consider

$$\begin{array}{ccc} E & \xrightarrow{g} & x_0 \times I \xrightarrow{r} x_0 \\ j \uparrow & \nearrow f & \\ B & & \end{array}$$

where $f(b) = (x_0, \phi(b))$ and $g(e) = (x_0, 0)$. By hypothesis, there exists a $g': E \rightarrow x_0 \times I$ such that $r \circ g' = r \circ g$ and $g' \circ j = f$, i.e., $g'(e) = (x_0, \psi(e))$ and $\psi \circ j = \phi$.

Conversely, assume that the property is satisfied for $j: B \rightarrow E$, and consider any diagram

$$\begin{array}{ccc} E & \xrightarrow{g} & X \times I \xrightarrow{r} X \\ j \uparrow & \nearrow f & \\ B & & \end{array}$$

where $r \circ f = r \circ g \circ j$. Using the projections on each factor, write $f(b) = (f_x(b), f_I(b))$ and $g(e) = (g_x(e), g_I(e))$; the commutativity requirement assures $f_x(b) = g_x(jb)$; but $f_I(b) = \psi j(b)$ for suitable $\psi: E \rightarrow I$, so that $g'(e) = (g_x(e), \psi(e))$ is the desired map.

In particular, if $j: B \rightarrow E$ is inclusion onto a retract of E , then j is a \mathfrak{B} -cofibration. Moreover,

4.3. If B is a functional Hausdorff space⁽¹³⁾, then each \mathfrak{B} -cofibration $j: B \rightarrow E$ is injective.

Proof. Let $j: B \rightarrow E$ be a map such that $j(b_0) = j(b_1)$ for some $b_0 \neq b_1$. There exists a map $\phi: B \rightarrow I$ with $\phi(b_0) = 0$, $\phi(b_1) = 1$; and ϕ cannot factor as $\phi = \psi \circ j$.

It is interesting to note that, within this framework, $\{\mathfrak{F}\text{-fibrations}\} \equiv \{\text{Hurewicz fibrations}\}$ has $\{\mathfrak{F}\text{-cofibrations}\} \equiv \{\text{all morphisms}\}$ for dual; and that $\{\mathfrak{B}\text{-fibrations}\} \equiv \{\text{all morphisms}\}$ has $\{\mathfrak{B}\text{-cofibrations}\} \equiv (\text{a restricted class of morphisms})$ for dual.

For a covariant functor $\Phi: \mathcal{K} \rightarrow \mathcal{L}$ the notion of Φ -cofibration is defined, in the customary manner, to be that of $\mathfrak{M}(\Phi)$ -cofibration. For contravariant functors, one uses the dual category, in which fibrations are exchanged against cofibrations.

5. Weak \mathfrak{M} -fibrations. We recall that, in the category Top, a map $p: E \rightarrow B$ is called a Dold fibration (= a map with the WCHP in [2]) if for each commutative diagram

⁽¹³⁾ I.e., for each pair $b_0 \neq b_1$ of points in B , there exists a continuous $\phi: B \rightarrow I$ such that $\phi(b_0) = 0$ and $\phi(b_1) = 1$.

$$\begin{array}{ccc} X & \xrightarrow{f} & E \\ i_0 \downarrow & & \downarrow p \\ X \times I & \xrightarrow{F} & B \end{array}$$

there exists a map $\tilde{F}: X \times I \rightarrow E$ such that $p \circ \tilde{F} = F$ and $\tilde{F} \circ i_0$ is fiber-homotopic to \tilde{f} .

For any $X \in \text{Top}$, let $\rho = \rho_X: X \times I \rightarrow X \times I$ denote the map

$$\begin{aligned} \rho(x, t) &= (x, 0), & 0 \leq t \leq \frac{1}{2}, \\ &= (x, 2t-1), & \frac{1}{2} \leq t \leq 1, \end{aligned}$$

and let $\mathfrak{R} = \{\rho_X \mid X \in \text{Top}\}$. Then

5.1. LEMMA. *A map $p: E \rightarrow B$ is a Dold fibration if and only if for each diagram*

$$\begin{array}{ccccc} X & \xrightarrow{i_0} & X \times I & \xrightarrow{g} & E \\ & & \searrow F & & \downarrow p \\ & & & & B \end{array}$$

with $p \circ g \circ i_0 = F \circ i_0$, there exists a $g': X \times I \rightarrow E$ such that $g \circ i_0 = g' \circ i_0$ and $p \circ g' = F \circ \rho$.

Because the proof of 5.1 is entirely analogous to that of 3.3, it is omitted.

Let \mathcal{K} be an arbitrary category and \mathfrak{M} be any family of its morphisms. With 5.1 in mind, we make the following

5.2. DEFINITION. A morphism $p: E \rightarrow B$ in \mathcal{K} is called a weak \mathfrak{M} -fibration if for each diagram

$$\begin{array}{ccccc} W & \xrightarrow{\mu} & X & \xrightarrow{g} & E \\ & & \searrow f & & \downarrow p \\ & & & & B \end{array}$$

in which $\mu \in \mathfrak{M}$ and $p \circ g \circ \mu = f \circ \mu$, there exists a $g': X \rightarrow E$ in \mathcal{K} , and an $r: X \rightarrow X$ in \mathfrak{M} such that $r \circ \mu = \mu$, $g \circ \mu = g' \circ \mu$, and $p \circ g' = f \circ r$.

REMARKS. (1) If \mathfrak{M} is such that $\mathfrak{M} \cap \mathcal{K}(X, X) = \{1_X: X \rightarrow X\}$ for each $X \in \mathcal{K}$, then $\{\text{weak } \mathfrak{M}\text{-fibrations}\} = \{\mathfrak{M}\text{-fibrations}\}$.

(2) If \mathfrak{M} is any class that contains all the identity morphisms, then each \mathfrak{M} -fibration is also a weak \mathfrak{M} -fibration.

(3) Unlike the \mathfrak{M} -fibrations, it is not in general true that $(\mathfrak{M} \subset \mathfrak{N}) \Rightarrow (\{\text{weak } \mathfrak{N}\text{-fibrations}\} \subset \{\text{weak } \mathfrak{M}\text{-fibrations}\})$.

5.3. DEFINITION. For any family \mathfrak{M} of morphisms in Top , we let $\mathfrak{M}_e = \mathfrak{M} \cup \{\text{all identities}\}$ and $\mathfrak{M}^* = \mathfrak{M}_e \cup \mathfrak{R}$. Then

5.4. THEOREM. *In the category Top ,*

- (a) $\{\text{weak } \mathfrak{K}_e\text{-fibrations}\} = \{\text{weak } \mathfrak{F}_e\text{-fibrations}\} = \{\text{Hurewicz fibrations}\};$
 (b) $\{\text{weak } \mathfrak{K}^*\text{-fibrations}\} = \{\text{weak } \mathfrak{F}^*\text{-fibrations}\} = \{\text{Dold fibrations}\}.$

Proof. (a) follows immediately from Remark (1) above and Theorems 3.3, 3.5. To prove (b): Observe that we have

$$\{\text{weak } \mathfrak{K}^*\text{-fibrations}\} \subset \{\text{weak } \mathfrak{F}^*\text{-fibrations}\}$$

because $\mathfrak{F} \subset \mathfrak{K}$ and all morphisms r that can possibly appear in Definition 5.2 are in \mathfrak{K}_e . The converse inclusion follows by an argument entirely analogous to that in the proof of 3.5. Finally, $\{\text{weak } \mathfrak{F}^*\text{-fibrations}\} = \{\text{Dold fibrations}\}$ by 5.1. This completes the proof (b).

Everything which has been said for weak \mathfrak{M} -fibrations can be dualized in the usual fashion, to yield weak \mathfrak{M} -cofibrations.

Appendix. In this section, we will give a proof of Theorem 1.1. We also include an example to show that in general $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$ and $E(\mathfrak{M}) \neq E(\hat{\mathfrak{M}})$.

Let \mathcal{K} be an arbitrary category, and let \mathfrak{M} be any subclass of its morphisms; the case that \mathfrak{M} is the family of all morphisms in \mathcal{K} is not excluded. We denote the elements of \mathfrak{M} by small Greek letters, α, β, \dots .

Given $A, B \in \text{ob}(\mathcal{K})$, by an \mathfrak{M} -word $m = (A, X_1, \dots, X_{2n-1}, B)$ from A to B we mean a finite chain of morphisms and objects

$$A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\alpha_2} X_3 \xrightarrow{f_2} \dots \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B$$

in which the \leftarrow and \rightarrow alternate, and each morphism α_i “in the wrong direction” either belongs to \mathfrak{M} or is an identity morphism. The class of all \mathfrak{M} -words from A to B is denoted by $\mathfrak{M}(A, B)$.

We introduce an equivalence relation \sim in $\mathfrak{M}(A, B)$ by

A.1. DEFINITION. Let $m, \hat{m} \in \mathfrak{M}(A, B)$. Declare $m \sim \hat{m}$ if there is a finite sequence $m = m_1, m_2, \dots, m_s = \hat{m}$ of elements of $\mathfrak{M}(A, B)$ in which each m_i is obtained from m_{i-1} (or from m_{i+1}) by one of the following two operations:

- (i) A pushout operation: replacement of a segment

$$\cdot \xrightarrow{f} X \xleftarrow{\alpha} Y \xrightarrow{g} Z \xleftarrow{\beta} \cdot$$

in m_{i-1} (or m_{i+1}) by

$$\cdot \xleftarrow{u \circ f} W \xrightarrow{v \circ \beta} \cdot$$

if there is a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{\alpha} & Y & \xrightarrow{g} & Z \\ & \searrow u & & \swarrow v & \\ & & W & & \end{array}$$

and $v \circ \beta \in \mathfrak{M}$ or is an identity morphism. This operation is denoted by

$$PO(X, Y, Z; u, W, v).$$

(ii) A pullback operation: replacement of a segment

$$\cdot \xleftarrow{\alpha} X \xrightarrow{f} Y \xleftarrow{\beta} Z \xrightarrow{g} \cdot$$

in m_{i-1} (or m_{i+1}) by

$$\cdot \xleftarrow{\alpha \circ u} W \xrightarrow{g \circ v} \cdot$$

if there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & Y & \xleftarrow{\beta} & Z \\ & \swarrow u & & \searrow v & \\ & W & & & \end{array}$$

and $\alpha \circ u \in \mathfrak{M}$ or is an identity morphism. This operation is denoted by

$$PB(X, Y, Z; u, W, v).$$

It is clear that \sim is an equivalence relation in $\mathfrak{M}(A, B)$; the equivalence class of m is denoted by $[m]$. Furthermore, the map $\mathfrak{M}(A, B) \otimes \mathfrak{M}(B, C) \rightarrow \mathfrak{M}(A, C)$ defined by

$$(A, X_1, \dots, X_{2n-1}, B) \otimes (B, Y_1, \dots, C) = (A, X_1, \dots, X_{2n-1}, B, Y_1, \dots, C)$$

is associative, and the equivalence class of $m \otimes m'$ depends only on that of m and of m' .

A.2. DEFINITION. The category \mathcal{X}/\mathfrak{M} is that having for objects: $\{A \mid A \in \text{ob}(\mathcal{X})\}$; for morphisms $\text{hom}(A, B)$: the equivalence classes⁽¹⁴⁾ in $\mathfrak{M}(A, B)$, with the composition law $\text{hom}(B, C) \circ \text{hom}(A, B) \rightarrow \text{hom}(A, C)$ given by $[m'] \circ [m] = [m \otimes m']$.

The identity $[e_A] \in \text{hom}(A, A)$ is the morphism $[A \xleftarrow{1} A \xrightarrow{1} A]$: for, given $[m] \in \text{hom}(B, A)$ we find that $m \otimes e_A$ is a word of form

$$\dots \xleftarrow{\quad} X \xrightarrow{f} A \xleftarrow{1} A \xrightarrow{1} A$$

and $PB(X, A, A; 1, X, f)$ shows $m \otimes e_A \sim m$; similarly, $e_A \otimes m' \sim m'$ for $m' \in \mathfrak{M}(A, B)$ so $[e_A]$ is the (unique) identity morphism of A .

A.3. Let each morphism in a word $m \in \mathfrak{M}(A, B)$ be either a member of \mathfrak{M} or an identity. Then $[m]$ is invertible in \mathcal{X}/\mathfrak{M} ; and its inverse is represented by m written in reverse order.

⁽¹⁴⁾ If \mathcal{X} is not a small category, $\text{hom}(A, B)$ need not be a set. However, in all the choices of \mathcal{X} and \mathfrak{M} considered in this paper, the class $\text{hom}(A, B)$ can be proved to be a set, so that the construction of this category is legitimate.

Proof. Letting m' be the word in reverse order, we have $m' \in \mathfrak{M}(B, A)$, and

$$m \otimes m' = A \xleftarrow{\alpha} X \xrightarrow{\beta} \dots \xleftarrow{\gamma} Y \xrightarrow{\gamma} B \xleftarrow{\gamma} Y \xrightarrow{\beta} \dots \xleftarrow{\alpha} X \xrightarrow{\alpha} A.$$

Starting from the middle with $PB(Y, B, Y; 1, Y, 1)$ and repeating, shows $m \otimes m' \sim A \xleftarrow{\alpha} X \xrightarrow{\alpha} A$; inserting $PO(A, X, A; 1, A, 1)$ we find $m \otimes m' \sim e_A$. Similarly $m' \otimes m \sim e_B$ and the proof is complete.

Observe that any word

$$A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\dots} X_{2n-2} \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B$$

is equivalent to

$$(A \xleftarrow{\alpha_1} X_1 \xrightarrow{1} X_1) \otimes (X_1 \xleftarrow{1} X_1 \xrightarrow{f_1} X_2) \otimes \dots \\ \otimes (X_{2n-2} \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{1} X_{2n-1}) \otimes (X_{2n-1} \xleftarrow{1} X_{2n-1} \xrightarrow{f_n} B).$$

Thus, each morphism in \mathcal{K}/\mathfrak{M} can be factored as $\hat{f}_n \circ \hat{\alpha}_n \circ \dots \circ \hat{f}_1 \circ \hat{\alpha}_1$ where, according to A.3, each $\hat{\alpha}_i$ is invertible.

The transformation $\eta: \mathcal{K} \rightarrow \mathcal{K}/\mathfrak{M}$ given by rules

$$\eta(A) = A \quad \text{on the objects,} \\ \eta(f) = [A \xleftarrow{1} A \xrightarrow{f} B] \quad \text{for each } f \in \mathcal{K}(A, B)$$

is easily seen to be a covariant functor, called the canonical projection. Because of A.3, $\eta(\alpha)$ is invertible in \mathcal{K}/\mathfrak{M} for each $\alpha \in \mathfrak{M}$; and the known [3] universality property of $(\eta, \mathcal{K}/\mathfrak{M})$, in a formulation adapted for our purposes, is

A.4. THEOREM. *Let $T: \mathcal{K} \rightarrow \mathcal{L}$ be any covariant functor to any category \mathcal{L} . Then*

(a) *There exists a covariant functor $\Delta: \mathcal{K}/\mathfrak{M} \rightarrow \mathcal{L}$ with $T = \Delta \circ \eta$ if and only if $T(\alpha)$ is invertible in \mathcal{L} for each $\alpha \in \mathfrak{M}$.*

(b) *If $T = \Delta \circ \eta$, then Δ is unique.*

Proof. (a) If $T = \Delta \circ \eta$ then, because $\eta(\alpha)$ is invertible for each $\alpha \in \mathfrak{M}$ and functors preserve invertible morphisms, T has the required property. Conversely, assume $T(\alpha)$ invertible for each $\alpha \in \mathfrak{M}$. Define Δ on the objects of \mathcal{K}/\mathfrak{M} by $\Delta(A) = T(A)$. For each word

$$m = A \xleftarrow{\alpha_1} X_1 \xrightarrow{f_1} X_2 \xleftarrow{\dots} X_{2n-2} \xleftarrow{\alpha_n} X_{2n-1} \xrightarrow{f_n} B$$

in $\mathfrak{M}(A, B)$, let

$$\Delta(m) = T(f_n) \circ T(\alpha_n)^{-1} \circ \dots \circ T(f_1) \circ T(\alpha_1)^{-1} \in \mathcal{L}(\Delta(A), \Delta(B))$$

where $T(\alpha_i)^{-1}$ is the inverse of $T(\alpha_i)$ in \mathcal{L} . If $m \sim m'$, then $\Delta(m) = \Delta(m')$ because the operations in changing m to m' are governed by commutative diagrams, and all

morphisms “in the wrong direction” belong to \mathfrak{M} . Thus, $[m] \mapsto \Delta(m)$ is a well-defined⁽¹⁵⁾ map $\Delta: \text{hom}(A, B) \rightarrow \mathcal{L}(\Delta(A), \Delta(B))$ for each $A, B \in \text{ob}(\mathcal{K}/\mathfrak{M})$ which with the above indicated correspondence of the objects, is easily seen to determine a covariant functor $\Delta: \mathcal{K}/\mathfrak{M} \rightarrow \mathcal{L}$. Clearly, $\Delta \circ \eta(A) = A$ on objects, and $\Delta \circ \eta(f) = \Delta[A \xleftarrow{1} A \xrightarrow{1} B] = T(f)$ for each $f \in \mathcal{K}(A, B)$, so $\Delta \circ \eta = T$ and the proof is complete.

(b) Assume $T = \Delta \circ \eta = \Gamma \circ \eta$; then $\Delta(A) = T(A) = \Gamma(A)$ for each $A \in \text{ob}(\mathcal{K}/\mathfrak{M})$. For each morphism of type $\hat{f} = [A \xleftarrow{1} A \xrightarrow{1} B]$ we have $\Delta(\hat{f}) = \Delta \circ \eta(f) = T(f) = \Gamma \circ \eta(f) = \Gamma(\hat{f})$. Each morphism of type $\hat{a} = [A \xleftarrow{\alpha} B \xrightarrow{1} B]$ is invertible in \mathcal{K}/\mathfrak{M} , with inverse $\hat{a}^{-1} = [B \xleftarrow{1} B \xrightarrow{\alpha} A]$. Thus $\Delta(\hat{a})$, $\Gamma(\hat{a})$ are invertible in \mathcal{L} and, from what we have already shown, $\Delta(\hat{a}) = \Delta(\hat{a}^{-1})^{-1} = \Gamma(\hat{a}^{-1})^{-1} = \Gamma(\hat{a})$. Since each morphism in \mathcal{K}/\mathfrak{M} is representable as a composition of morphisms of these two types, we find $\Delta[m] = \Gamma[m]$ for each morphism $[m]$ and the proof is complete.

REMARK. If \mathfrak{M} is the family of all morphisms in \mathcal{K} , then all the morphisms of \mathcal{K}/\mathfrak{M} are invertible therefore each $\text{hom}(X, X)$ is a group, $\pi(X)$. Furthermore, for each $\phi \in \text{hom}(X, Y)$ there is a homomorphism $\phi^*: \pi(X) \rightarrow \pi(Y)$ given by $\phi^*(f) = \phi \circ f \circ \phi^{-1}$, and clearly ϕ^* is an isomorphism. Thus the groups $\pi(X)$ belonging to a “component” of \mathcal{K} are all isomorphic; if \mathcal{K} is “connected” then, as in topology, we can define the fundamental group $\pi(\mathcal{K})$ of a category. From C2 it follows that a covariant functor $\Phi: \mathcal{K} \rightarrow \mathcal{L}$ on “connected” categories induces a homomorphism $\pi(\mathcal{K}) \rightarrow \pi(\mathcal{L})$. It is easy to see that for Top_0 , the category of based topological spaces with base-point preserving continuous maps, each word from (X, x_0) to (X, x_0) becomes equivalent to $(X, x_0) \leftarrow (x_0, x_0) \rightarrow (X, x_0)$ and therefore that $\pi(\text{Top}_0) = 0$. Similarly, it can be shown that $\pi(\text{Top}) = 0$.

We now show

A.5. PROPOSITION. *There exists a category \mathcal{K} and a class \mathfrak{M} such that $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$ and $E(\hat{\mathfrak{M}}) \neq E(\mathfrak{M})$, where $\hat{\mathfrak{M}}$ is the maximal stable subclass in \mathfrak{M} .*

Proof. Consider the following diagram

$$*_A \xrightarrow{h'} S_0^2 \xrightleftharpoons[\beta]{\alpha} S_1^2 \xrightarrow{h} *_B$$

where S_0^2, S_1^2 are two distinct 2-spheres, $*_A, *_B$ are distinct points, $h'(*_A)$ = north pole of S_0^2 , α is the rotation about the z -axis with angle π , and β is the rotation about the same axis with angle π/u , u an irrational. Let \mathcal{K} be the category with these four distinct objects, and with morphisms generated by h', α, β, h .

Let $\mathfrak{M} = \{h, h'\}$; then it is immediate that $\bar{\mathfrak{M}} = \text{all morphisms in } \mathcal{K} \text{ and } E(\mathfrak{M}) = \bar{\mathfrak{M}} - \mathfrak{M}$. Moreover

⁽¹⁵⁾ The set of all morphisms $A \rightarrow B$ in a category \mathcal{L} will be denoted by $\mathcal{L}(A, B)$; if $\mathcal{L} = \mathcal{K}/\mathfrak{M}$, this set is denoted by $\text{hom}(A, B)$.

(a) $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$. Let \mathcal{L} be the smallest subcategory of Top containing \mathcal{K} and the inverses $\bar{\alpha}, \bar{\beta}$ of α, β . Let $\lambda: \mathcal{K} \rightarrow \mathcal{L}$ be the inclusion functor. Since λ sends every member of $E(\mathfrak{M})$ to an isomorphism in \mathcal{L} , there exists (A.4) a functor $\Delta: \mathcal{K}/E(\mathfrak{M}) \rightarrow \mathcal{L}$ such that $\Delta \circ \eta = \lambda$. Now, $\alpha \notin E(E(\mathfrak{M}))$: for if $\eta(\alpha)\eta(g)=1$ for some g in \mathcal{K} , then $\lambda(\alpha \circ g)=1$ in \mathcal{L} , i.e., $\alpha \circ g=1$ in \mathcal{L} ; however α does not have a right inverse in \mathcal{K} , so such a g cannot exist in \mathcal{K} . Thus, $E(E(\mathfrak{M})) \neq E(\mathfrak{M})$. Moreover, since $\alpha \in E(\mathfrak{M}) \subset [E(\mathfrak{M})]^-$, this shows also that $E(\mathfrak{M})$ is not stable; furthermore, $\hat{\mathfrak{M}} = \emptyset$, since h and h' do not have inverses in \mathcal{K} , so we also have $E(\mathfrak{M}) \neq E(\hat{\mathfrak{M}})$.

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